Darboux–Egoroff metrics, rational Landau–Ginzburg potentials and the Painlevé VI equation

2003 J. Phys. A: Math. Gen. 36 1013

(http://iopscience.iop.org/0305-4470/36/4/311)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 1013–1028 PII: S0305-4470(03)53840-9

Darboux–Egoroff metrics, rational Landau–Ginzburg potentials and the Painlevé VI equation

H Aratyn[1], **J F Gomes**[2] **and A H Zimerman**[2]

[1] Department of Physics, University of Illinois at Chicago, 845 W Taylor St, Chicago, IL 60607-7059, USA
[2] Instituto de Física Teórica–UNESP, Rua Pamplona 145, 01405-900 São Paulo, Brazil

E-mail: aratyn@uic.edu, jfg@ift.unesp.br and zimerman@ift.unesp.br

Received 23 September 2002
Published 15 January 2003
Online at stacks.iop.org/JPhysA/36/1013

Abstract
We present a class of three-dimensional integrable structures associated with the Darboux–Egoroff metric and classical Euler equations of free rotations of a rigid body. They are obtained as canonical structures of rational Landau–Ginzburg potentials and provide solutions to the Painlevé VI equation.

PACS numbers: 02.30.Ik, 02.30.Jr, 02.40.Vh

1. Introduction

The flat coordinates and prepotentials play a prominent role in the study and classification of solutions to the Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations [1–4]. Our interest is in finding the formalism which would provide a universal approach to canonical integrable structures behind the WDVV equations. Here, we understand canonical integrable structures as hierarchies formulated in terms of canonical coordinates and with evolution flows obtained in a conventional way from the Riemann–Hilbert factorization problem with twisting condition imposed [5, 6]. This approach coincides with the framework of the Darboux–Egoroff metric constructed in terms of the Lamé coefficients and their symmetric rotation coefficients [3]. Additional reduction (scaling invariance) effectively turns the evolution flows into isomonodromic deformations (up to the Laplace transformation) and allows the flow evolution equations to be rewritten as Schlesinger equations [3]. One can associate a τ function with the factorization problem which, upon imposition of the conformal condition, coincides with the corresponding isomonodromic τ function [7, 8]. The τ function appears to be a key object in this construction due to the fact that all the other objects of canonical formalism (e.g. rotation coefficients) can be obtained from it. The conformal condition introduces a notion of scaling dimension or homogeneity of the τ function which is convenient to use in order to classify integrable models. For the integer scaling dimensions of the τ function the integrable hierarchies can be reproduced by the (C)KP-like formalism [8, 9]. The τ functions can be calculated in the Grassmannian approach to the multi-component KP hierarchy [7, 10]. The structures with the τ functions possessing fractional scaling dimension lack such a clear underlying foundation. An exception to this rule is the trivial two-dimensional case where the τ function can be explicitly found for all models.

This work studies three-dimensional models with the τ functions having the scaling dimension equal to the fractional number of $1/4$ in the hope of providing data for its future general formalism. The integrable structures are obtained from Frobenius manifolds associated with a class of rational Lax functions [11]. The construction of canonical coordinates for these polynomials generalizes the well-known construction of canonical coordinates associated with the monic polynomials [3, 12]. It is worth noting that the space of monic polynomials endowed with a natural metric provides one of the most standard examples of the Frobenius manifolds.

We find a closed expression for the τ function for one of the considered models. We also associate the canonical structure with algebraic solutions for the Painlevé VI equations considered by Hitchin [13–15] and Segert [16, 17].

The paper is organized as follows. In section 2, we define the canonical integrable structure behind the WDVV equations emphasizing its connection to the Darboux–Egoroff metric. The τ function plays a key role in this formalism. In this section we also introduce homogeneity as a scaling dimension. In addition, we discuss the relation of canonical integrable structures to the flat coordinates, structure constants and associativity equations.

The special case of three dimensions is presented in section 3 where the Darboux–Egoroff equations are shown to take the form of the classical Euler equations of free rotations of a rigid body. Also, the scaling dimension of the τ function is found to be related to the integral of the Euler equations.

The rational potentials and their metric along with the associated canonical coordinates related to the Darboux–Egoroff metric are described in section 4. Examples of the three-dimensional canonical integrable models derived from the rational potentials are given in section 5 and the relation to the Painlevé VI equation is established. The Darboux–Egoroff metric is described in detail for the discussed three-dimensional models. The explicit form of the τ function is found for one of the models. Section 6 presents concluding remarks.

2. From factorization problem to Darboux–Egoroff metric

Let the 'bare' wave (matrix) function be defined in terms of canonical coordinates $\mathbf{u} = (u_1, \ldots, u_N)$ as

$$\Psi_0(\mathbf{u}, z) = \exp\left(z\sum_{j=1}^{N} E_{jj}u_j\right) = \mathrm{e}^{zU} \tag{2.1}$$

where $U = \mathrm{diag}(u_1, \ldots, u_N)$ and the unit matrix E_{jj} has the matrix elements $(E_{jj})_{kl} = \delta_{kj}\delta_{lj}$.

Define the 'dressed' wave matrix Ψ:

$$\Psi(\mathbf{u}, z) = \Theta(\mathbf{u}, z)\Psi_0(\mathbf{u}, z) \tag{2.2}$$

obtained by acting on the 'bare' wave (matrix) function $\Psi_0(\mathbf{u}, z)$ with the dressing matrix

$$\Theta(\mathbf{u}, z) = 1 + \theta^{(-1)}(\mathbf{u})z^{-1} + \theta^{(-2)}z^{-2} + \cdots. \tag{2.3}$$

The 'dressed' wave matrix Ψ enters a factorization problem:

$$\Psi(\mathbf{u}, z)g(z) = \Theta(\mathbf{u}, z)\Psi_0(\mathbf{u}, z)g(z) = M(\mathbf{u}, z) \tag{2.4}$$

where g is a map from $z \in S^1$ to the Lie group G and $M(\mathbf{u}, z)$ is a positive power series in z:

$$M(\mathbf{u}, z) = M_0(\mathbf{u}) + M_1(\mathbf{u})z + M_2(\mathbf{u})z^2 + \cdots. \tag{2.5}$$

If the map $g(z)$ is such that it satisfies the twisting condition $g^{-1}(z) = g^T(-z)$ then it follows that the matrices $\Theta(\mathbf{u}, z)$ and $M(\mathbf{u}, z)$ from equation (2.4) will also satisfy the twisting conditions $\Theta^T(\mathbf{u}, -z) = \Theta^{-1}(\mathbf{u}, z)$, $M^T(\mathbf{u}, -z) = M^{-1}(\mathbf{u}, z)$ [5, 6]. In this case, the matrix $\theta^{(-1)}(\mathbf{u})$ from expansion (2.3) must be symmetric, $\theta^{(-1)}(\mathbf{u}) = (\theta^{(-1)})^T(\mathbf{u})$ and the matrix $M_0(\mathbf{u})$ from expression (2.5) must be orthogonal, $M_0^{-1}(\mathbf{u}) = M_0^T(\mathbf{u})$.

The factorization problem (2.4) leads to the flow equations

$$\frac{\partial}{\partial u_j}\Theta(\mathbf{u}, z) = -(\Theta z E_{jj}\Theta^{-1})_-\Theta(\mathbf{u}, z) \tag{2.6}$$

$$\frac{\partial}{\partial u_j}M(\mathbf{u}, z) = (\Theta z E_{jj}\Theta^{-1}t)_+M(\mathbf{u}, z) \tag{2.7}$$

where $(\ldots)_\pm$ denote the projections on positive/negative powers of z. Due to equation (2.6) the one-form $\mathrm{Res}_z(\mathrm{tr}(\Theta^{-1}z(\mathrm{d}\Theta/\mathrm{d}z)\,\mathrm{d}U))$ is closed. Conventionally, one associates with this one-form a τ function through

$$d\log\tau = \mathrm{Res}_z\left(\mathrm{tr}\left(\Theta^{-1}z\frac{\mathrm{d}\Theta}{\mathrm{d}z}\,\mathrm{d}U\right)\right) \tag{2.8}$$

where $d = \sum_{j=1}^N \partial_j\,\mathrm{d}u_j$ with $\partial_j = \partial/\partial u_j$. One can verify that this definition of the τ function implies the relation $\theta_{ii}^{(-1)} = -\partial_i\log\tau$ for the diagonal elements of the $\theta^{(-1)}$ matrix. We will parametrize the $\theta^{(-1)}$ matrix as follows:

$$\theta_{ij}^{(-1)} = \begin{cases}\beta_{ij} & i \neq j\\ -\partial_i\log\tau & i = j\end{cases}. \tag{2.9}$$

Due to equation (2.6) the symmetric off-diagonal components β_{ij} satisfy

$$\partial_j\beta_{ik} = \beta_{ij}\beta_{jk} \qquad i, j, k \text{ distinct} \tag{2.10}$$

It follows from the definition (2.2) that the 'dressed' wave matrix Ψ will satisfy flow equations identical to those in (2.7):

$$\frac{\partial}{\partial u_j}\Psi(\mathbf{u}, z) = (\Theta z E_{jj}\Theta^{-1})_+\Psi(\mathbf{u}, z). \tag{2.11}$$

The projection in (2.11) contains only two terms: $(\Theta z E_{jj}\Theta^{-1})_+ = zE_{jj} + V_j$ where $V_j \equiv [\theta^{(-1)}, E_{jj}]$. Accordingly,

$$\frac{\partial\Psi}{\partial u_j} = (zE_{jj} + V_j)\Psi \tag{2.12}$$

and $\partial\Theta/\partial u_j = z[E_{jj}, \Theta] + V_j\Theta$. Projecting on diagonal directions yields $\partial_i\partial_j\log\tau = -\beta_{ij}^2$ for $i \neq j$. Hence the dressing matrix can be expressed in terms of one object only, the τ function.

We now define Euler and identity vectorfields E and I as

$$E = \sum_{i=1}^N u_i\frac{\partial}{\partial u_i} \qquad I = \sum_{i=1}^N \frac{\partial}{\partial u_i}. \tag{2.13}$$

Their action on Ψ can easily be found from (2.12) by summing over j:

$$E(\Psi) = (zU + V)\Psi \qquad I(\Psi) = z\Psi \tag{2.14}$$

where we introduced matrix $V = [\theta^{(-1)}, U]$ with the components

$$V_{ij} = (u_j - u_i)\theta_{ij}^{(-1)} = (u_j - u_i)\beta_{ij} \qquad i, j = 1, \ldots, N. \tag{2.15}$$

Similarly, $E(\Theta) = z[U, \Theta] + V\Theta$.

Note that $E(\Psi_0) = z\,\mathrm{d}\Psi_0/\mathrm{d}z$. We will require that the same equality holds for the action of the grading operator $z\,\mathrm{d}/\mathrm{d}z$ and of the Euler operator E on Ψ:

$$z\frac{\mathrm{d}}{\mathrm{d}z}\Psi = E(\Psi) = (zU + V)\Psi. \tag{2.16}$$

Relation (2.16) is called the conformal condition [3]. It holds provided $E(\Theta) = z\,\mathrm{d}\Theta/\mathrm{d}z$. Also, from equation (2.6) we get by summing over all indices j that $I(\Theta) = 0$. In particular, for $\theta^{(-1)}$ we get

$$E(\beta_{ij}) = -\beta_{ij} \qquad I(\beta_{ij}) = 0 \tag{2.17}$$

$$E(\partial_i \log\tau) = -\partial_i \log\tau \qquad I(\partial_i \log\tau) = 0. \tag{2.18}$$

Since $E\partial_i = \partial_i E - \partial_i$ we see that $E \log\tau = \text{constant}$ or $E(\tau) = \text{constant}\,\tau$ is compatible with the conformal condition (2.18). This constant defines the scaling dimension (or homogeneity) of the τ function.

Plugging relation $E\Theta = z\,\mathrm{d}\Theta/\mathrm{d}z$ into the formula (2.8) for the τ function one obtains [9]

$$\partial_j \log\tau = \mathrm{Res}_z(\mathrm{tr}(\Theta^{-1}E\Theta E_{jj})) = \tfrac{1}{2}\,\mathrm{tr}(V_j V). \tag{2.19}$$

Comparing with expression for the isomonodromy τ function τ_I [3] one concludes that $\tau_I = 1/\sqrt{\tau}$ [7, 8]. The isomonodromy τ function τ_I gives the Hamiltonian formulation $\partial_j V = \{H_j, V\}$ for the equation

$$\partial_j V = [V_j, V] \tag{2.20}$$

via the formula $\partial_j \log\tau_I = H_j$. Equation (2.20) follows from the compatibility of equations (2.12), (2.14) and (2.16). One clearly sees from (2.19) and (2.20) that

$$I(\log\tau) = 0 \qquad I(V) = 0 \tag{2.21}$$

since $\sum_{j=1}^N V_j = 0$.

The similarity transformation $V \to \mathcal{V} = M_0^{-1} V M_0$ transforms V to the constant matrix $\mathcal{V}$ ($\partial_j \mathcal{V} = 0$) due to the flow equations $\partial_j M_0 = V_j M_0$, which follow from (2.7). Assume now that there exists an invertible matrix S which diagonalizes $\mathcal{V}$:

$$S^{-1}\mathcal{V}S = \mu = \sum_{j=1}^N \mu_j E_{jj} \tag{2.22}$$

where μ is the constant diagonal matrix. Next, define a matrix

$$M(u) = M_0(u)S = (m_{ij}(u))_{1\leqslant i,j\leqslant N} \tag{2.23}$$

which governs transformation from the canonical coordinates u_j, $j = 1, \ldots, N$ to the flat coordinates x^α, $\alpha = 1, \ldots, N$. Let constant non-degenerate metric be given by the matrix

$$\eta = (\eta_{\alpha\beta})_{1\leqslant\alpha,\beta\leqslant N} = M^T M = S^T S \qquad \text{and denote } \eta^{-1} = (\eta^{\alpha\beta})_{1\leqslant\alpha,\beta\leqslant N} \tag{2.24}$$

hence $\eta_{\alpha\beta} = \sum_{i=1}^N m_{i\alpha} m_{i\beta}$. Then the derivatives with respect to the flat coordinates x^α, $\alpha = 1, \ldots, N$ are given by

$$\frac{\partial}{\partial x^\alpha} = \sum_{i=1}^N \frac{m_{i\alpha}}{m_{i1}} \frac{\partial}{\partial u_i} \tag{2.25}$$

with the reversed relation being

$$\frac{\partial}{\partial u_i} = \sum_{\alpha,\beta=1}^N \eta^{\alpha\beta} m_{i1} m_{i\beta} \frac{\partial}{\partial x^\alpha}. \tag{2.26}$$

The structure constants

$$c_{\alpha\beta\gamma} = \sum_{i=1}^{N} \frac{m_{i\alpha} m_{i\beta} m_{i\gamma}}{m_{i1}} \tag{2.27}$$

satisfy the associativity equation

$$\sum_{\delta\gamma=1}^{N} c_{\alpha\beta\delta}\eta^{\delta\gamma} c_{\gamma\sigma\rho} = \sum_{\delta\gamma=1}^{N} c_{\alpha\sigma\delta}\eta^{\delta\gamma} c_{\gamma\beta\rho} \qquad \alpha, \beta, \sigma = 1, \ldots, N \tag{2.28}$$

and are given by derivations of the prepotential F:

$$c_{\alpha\beta\gamma} = \frac{\partial^3 F}{\partial x^\alpha \partial x^\beta \partial x^\gamma}. \tag{2.29}$$

The metric $g = \sum_{\alpha,\beta=1}^{N} \eta_{\alpha\beta}\, \mathrm{d}x^\alpha\, \mathrm{d}x^\beta$ equals in terms of the canonical coordinates

$$g = \sum_{i=1}^{N} h_i^2 (\mathrm{d}u_i)^2 \tag{2.30}$$

with Lamé coefficients $h_i = m_{i1}$ being such that the corresponding rotation coefficients

$$\beta_{ij} = \frac{1}{h_j}\frac{\partial h_i}{\partial u_j} \tag{2.31}$$

satisfy conditions of the Darboux–Egoroff metric, namely $\beta_{ij} = \beta_{ji}$ together with the relations (2.10) and $I(\beta_{ij}) = 0$. Note that the Darboux–Egoroff condition $\beta_{ij} = \beta_{ji}$ is equivalent to square of the Lamé coefficient being a gradient of some potential ϕ: $h_i^2 = \partial\phi/\partial u_i$.

Furthermore, the Euler operator is given in terms of the flat coordinates as

$$E = \sum_{\alpha=1}^{N} (d_\alpha x^\alpha + r_\alpha)\frac{\partial}{\partial x^\alpha} \tag{2.32}$$

with $d_\alpha r_\alpha = 0$ and $d_\alpha = 1 + \mu_1 - \mu_\alpha$ [3]. The quasi-homogeneity condition states that

$$E(F) = d_F F + \text{quadratic terms} \tag{2.33}$$

where the number d_F denotes the degree of the prepotential F.

Following [3, 17], we will say that a function ψ is of 'homogeneity c' or 'scaling dimension c' or 'E-degree', if $E(\psi) = c\psi$ for a constant c.

For the Lamé coefficients h_i with $\sum_{i=1}^{N} h_i^2 = \sum_{i=1}^{N} m_{i1}^2 = \eta_{11}$ the homogeneity must be zero for the constant metric tensor with $\eta_{11} \neq 0$ since

$$0 = E\left(\sum_{i=1}^{N} h_i^2\right) = \sum_{i=1}^{N} 2h_i c h_i = 2\eta_{11} c. \tag{2.34}$$

From relation $E(\eta_{\alpha\beta}) = (d_F - d_1 - d_\alpha - d_\beta)\eta_{\alpha\beta}$, obtained by acting with the Euler vectorfield on both sides of (2.29), it follows that $d_F = 3d_1$ for $\eta_{11} \neq 0$ and if $d_F \neq 3d_1$ then $\eta_{11} = 0$. Hence the value of the homogeneity of the prepotential indicates when the homogeneity of the Lamé coefficients vanishes.

For the class of models we consider here the Lamé coefficients h_i are given by the formula

$$h_i^2 = \frac{\partial x_1}{\partial u_i} \tag{2.35}$$

which agrees with a general feature of Frobenius manifolds endowed with the invariant metric [3]. Accordingly, the homogeneity of the Lamé coefficients is a constant number equal to $\big(\sum_{\alpha=1}^{N} d_\alpha \eta^{\alpha 1} - 1\big)$ according to

$$E\big(h_i^2\big) = E\partial_i(x_1) = \partial_i E(x_1) - \partial_i(x_1) = \left(\sum_{\alpha=1}^{N} d_\alpha \eta^{\alpha 1} - 1\right) h_i^2. \tag{2.36}$$

Hence, for $\eta_{11} \neq 0$ it holds that $\sum_{\alpha=1}^{N} d_\alpha \eta^{\alpha 1} = 1$.

Note also that $E(h_i) = ch_i$ is consistent with the conformal condition in (2.17) for the arbitrary constant homogeneity c.

For the Darboux–Egoroff metric the identity vectorfield vanishes when acting on the Lamé coefficient as follows from

$$I(h_i) = \sum_{j=1}^{N} \beta_{ij} h_j = \sum_{j=1}^{N} \frac{1}{h_i} \frac{\partial h_j}{\partial u_i} h_j = \frac{1}{2h_i} \frac{\partial \eta_{11}}{\partial u_i} = 0. \tag{2.37}$$

3. The three-dimensional case

Let us now consider the three-dimensional manifolds. In this case, we can rewrite the antisymmetric matrix V as

$$V = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix} \tag{3.1}$$

or $(V)_{ij} = (u_j - u_i)\beta_{ij} = \epsilon_{ijk}\omega_k$. From (2.17) and (2.21) we see that ω_k vanishes when acted on by the vectorfields E and I. That makes ω_k effectively a function of one variable s such that $E(s) = I(s) = 0$. Let us choose

$$s = \frac{u_2 - u_1}{u_3 - u_1}. \tag{3.2}$$

Then equation (2.20) takes a form equivalent to the Euler top equations

$$\frac{\mathrm{d}\omega_1}{\mathrm{d}s} = \frac{\omega_2\omega_3}{s} \tag{3.3}$$

$$\frac{\mathrm{d}\omega_2}{\mathrm{d}s} = \frac{\omega_1\omega_3}{s(s-1)} \tag{3.4}$$

$$\frac{\mathrm{d}\omega_3}{\mathrm{d}s} = \frac{\omega_1\omega_2}{1-s}. \tag{3.5}$$

One verifies that $\mathrm{d}\big(\sum_{k=1}^{3} \omega_k^2\big)/\mathrm{d}s = 0$. Consequently,

$$\sum_{k=1}^{3} \omega_k^2 = -R^2 \tag{3.6}$$

with R being a constant is the integral of equations (3.3)–(3.5). The same constant R characterizes the homogeneity of the τ function, as we will show now. Starting from expression (2.19) one finds for the scaling dimension [18]

$$E(\log\tau) = \frac{1}{2}\sum_{j=1}^{3} u_j \operatorname{tr}(V_j V) = \frac{1}{2}\operatorname{tr}(V^2) = \frac{1}{2}\operatorname{tr}(\mu^2) = \frac{1}{2}\sum_{\alpha=1}^{3} \mu_\alpha^2. \tag{3.7}$$

Recalling that $(V)_{ij} = \epsilon_{ijk}\omega_k$ we can rewrite the above as

$$E(\log\tau) = \frac{1}{2}\sum_{j=1}^{3}\sum_{i=1}^{3}(\epsilon_{ijk}\omega_k)^2 = -\sum_{k=1}^{3}\omega_k^2 = R^2 \tag{3.8}$$

and since $\mu_\alpha = 1 - d_\alpha + d/2$ with $d = d_F - 3$

$$R^2 = \sum_{\alpha=1}^{3}\left(\frac{1}{2} - d_F/2 + d_\alpha\right)^2. \tag{3.9}$$

We have seen above that for η_{11} different from zero the homogeneity of the Lamé coefficients h_i must vanish. In such a case, the Lamé coefficients h_i depend only on one variable s due to the fact that $I(h_i) = E(h_i) = 0$. The relations $\partial_j h_i^2 = \partial_i h_j^2$ translate for the function $h_i^2(s)$ to

$$s\frac{\mathrm{d}h_1^2}{\mathrm{d}s} = (s-1)s\frac{\mathrm{d}h_2^2}{\mathrm{d}s} = (1-s)\frac{\mathrm{d}h_3^2}{\mathrm{d}s}. \tag{3.10}$$

Also, since

$$\omega_k = \frac{u_j - u_i}{2h_i h_j}\frac{\partial h_i^2}{\partial u_j} \qquad i, j, k \text{ cyclic} \tag{3.11}$$

we find, e.g.

$$\omega_3 = \frac{s}{2h_1h_2}\frac{\mathrm{d}h_1^2}{\mathrm{d}s} \qquad \omega_2 = \frac{s}{2h_1h_3}\frac{\mathrm{d}h_1^2}{\mathrm{d}s} \tag{3.12}$$

and so $h_3\omega_2 = h_2\omega_3$ and similarly $h_1\omega_2 = h_2\omega_1$. We conclude that

$$\omega_i^2 = -\frac{R^2}{\eta_{11}}h_i^2 \qquad i = 1, 2, 3 \tag{3.13}$$

and comparing equations (3.3)–(3.5) with equation (3.10) we obtain as in [17]

$$s\frac{\mathrm{d}h_1^2}{\mathrm{d}s} = (s-1)s\frac{\mathrm{d}h_2^2}{\mathrm{d}s} = (1-s)\frac{\mathrm{d}h_3^2}{\mathrm{d}s} = -2\mathrm{i}\frac{R}{\sqrt{\eta_{11}}}h_1h_2h_3. \tag{3.14}$$

4. Rational Landau–Ginsburg models

Following Aoyama and Kodama [11] we study a rational potential

$$W(z) = \frac{1}{n+1}z^{n+1} + a_{n-1}z^{n-1} + \cdots + a_0 + \frac{v_1}{z - v_{m+1}} + \frac{v_2}{2(z - v_{m+1})^2} + \cdots + \frac{v_m}{m(z - v_{m+1})^m} \tag{4.1}$$

which is known to characterize the topological Landau–Ginzburg (LG) theory. The rational potential in this form can be regarded as the Lax operator of a particular reduction of the dispersionless KP hierarchy [11, 19, 20].

The space of rational potentials from (4.1) is naturally endowed with the metric

$$g(\partial_t W, \partial_{t'} W) = \mathrm{Res}_{z\in\mathrm{Ker}W'}\left(\frac{\partial_t W \partial_{t'} W}{W'}\right)\mathrm{d}z \tag{4.2}$$

where $\partial_t W = \partial_t a_{n-1}z^{n-1} + \cdots + \partial_t a_0 + \frac{\partial_t v_1}{z - v_{m+1}} + \cdots$ describes a tangent vector to the space of rational potentials obtained by taking the derivative of all coefficients with respect to their argument. $W'(z)$ is a derivative with respect to z of the rational potential W:

$$W'(z) = z^n + (n-1)a_{n-1}z^{n-2} + \cdots - \frac{v_m}{(z - v_{m+1})^{m+1}}. \tag{4.3}$$

Next, we find the flat coordinates $x_\alpha, \alpha = 1, \ldots, m+1$ and $\tilde{x}_\gamma, \gamma = 1, \ldots, n$ such that

$$g\left(\frac{\partial W}{\partial x_\alpha}, \frac{\partial W}{\partial x_\beta}\right) = \eta_{\alpha\beta} \qquad g\left(\frac{\partial W}{\partial \tilde{x}_\gamma}, \frac{\partial W}{\partial \tilde{x}_\delta}\right) = \tilde{\eta}_{\gamma\delta} \qquad g\left(\frac{\partial W}{\partial x_\alpha}, \frac{\partial W}{\partial \tilde{x}_\gamma}\right) = 0 \tag{4.4}$$

with constant and non-degenerate matrices $\eta_{\alpha\beta}$ and $\tilde{\eta}_{\gamma\delta}$.

Consider first the function $w = w(W, z)$ such that $W(z) = w^{-m}/m$ and $z = x_{m+1} + x_m w + \cdots + x_1 w^m = \sum_{\alpha=1}^{m+1} x_\alpha w^{m+1-\alpha}$. We take $z \sim x_{m+1}$ or $|w| \ll 1$. It follows that

$$W'\,\mathrm{d}z = -\frac{1}{w^{m+1}}\,\mathrm{d}w \qquad \frac{\partial W}{\partial x_\alpha} = W'\frac{\partial z}{\partial x_\alpha} = W' w^{m+1-\alpha}. \tag{4.5}$$

Consequently,

$$\begin{aligned} g\left(\frac{\partial W}{\partial x_\alpha}, \frac{\partial W}{\partial x_\beta}\right) &= -\mathrm{Res}_{z=\infty}\left(\frac{(\partial W/\partial x_\alpha)(\partial W/\partial x_\beta)}{W'}\right)\mathrm{d}z \\ &= -\mathrm{Res}_{z=\infty}(W' w^{m+1-\alpha} w^{m+1-\beta})\,\mathrm{d}z \\ &= \mathrm{Res}_{w=\infty}\left(\frac{w^{m+1-\alpha} w^{m+1-\beta}}{w^{m+1}}\right)\mathrm{d}w = \delta_{\alpha+\beta=m+2}. \end{aligned} \tag{4.6}$$

Hence x_α are flat coordinates with the metric $\eta_{\alpha\beta} = \delta_{\alpha+\beta=m+2}$. The coefficients v_j, $j = 1, \ldots, m+1$ of $W(z)$ are given in terms of the flat coordinates as [11]

$$\begin{aligned} v_k &= \sum_{\alpha_1+\cdots+\alpha_k=(k-1)m+k} x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_k} \qquad k = 1, \ldots, m \\ v_{m+1} &= x_{m+1}. \end{aligned} \tag{4.7}$$

Examples are

$$v_m = (x_m)^m, v_{m-1} = (m-1)x_{m-1}(x_m)^{m-2}, \ldots, v_1 = x_1. \tag{4.8}$$

To represent the remaining coefficients of a_i, $i = 1, \ldots, n$ of W in terms of the flat coordinates we consider a relation

$$z = w + \frac{\tilde{x}_1}{w} + \frac{\tilde{x}_2}{w^2} + \cdots + \frac{\tilde{x}_n}{w^n} \tag{4.9}$$

valid for large z and $|w| \gg 1$. In this limit we impose a relation $W = w^{n+1}/(n+1)$ from which it follows that

$$W'\,\mathrm{d}z = w^n\,\mathrm{d}w \qquad \frac{\partial W}{\partial \tilde{x}_\gamma} = W'\frac{\partial z}{\partial \tilde{x}_\gamma} = W' w^{-\gamma}. \tag{4.10}$$

We find

$$\begin{aligned} g\left(\frac{\partial W}{\partial \tilde{x}_\gamma}, \frac{\partial W}{\partial \tilde{x}_\delta}\right) &= \mathrm{Res}_{z\in\mathrm{Ker}W'}\left(\frac{(\partial W/\partial \tilde{x}_\gamma)(\partial W/\partial \tilde{x}_\delta)}{W'}\right)\mathrm{d}z \\ &= \mathrm{Res}_{z\in\mathrm{Ker}W'}(W' w^{-\gamma} w^{-\delta})\,\mathrm{d}z = \mathrm{Res}_{w=0} w^{n-\gamma-\delta}\,\mathrm{d}w = \delta_{\gamma+\delta=n+1}. \end{aligned} \tag{4.11}$$

Hence $\tilde{x}_\gamma$ are flat coordinates with the metric $\tilde{\eta}_{\gamma\delta} = \delta_{\gamma+\delta=n+1}$. By similar considerations $\eta_{\alpha\gamma} = 0$ for $\alpha = 1, \ldots, m+1, \gamma = 1, \ldots, n$.

From expression (4.9) and $W(z) = w^{n+1}/(n+1)$ one can find relations between coefficients a_γ and $\tilde{x}_\gamma$ [11] starting with $a_{n-1} = -\tilde{x}_1$ and so on.

We will now show how to associate with the rational potentials W canonical coordinates u_i, $i = 1, \ldots, n+m+1$ for which the metric (4.2) becomes a Darboux–Egoroff metric.

Let α_i, $i = 1, \ldots, n+m+1$ be roots of the rational potential $W'(z)$ in (4.3). Equivalently, $W'(\alpha_i) = 0$ for all $i = 1, \ldots, n+m+1$. Thus $W'(z)$ can be rewritten as

$$W'(z) = \frac{\prod_{j=1}^{n+m+1}(z - \alpha_j)}{(z - v_{m+1})^{m+1}}. \tag{4.12}$$

Next, define the canonical coordinates as

$$u_i = W(\alpha_i) \qquad i = 1, \ldots, n+m+1. \tag{4.13}$$

The identity

$$\delta^i_j = \frac{\partial u_i}{\partial u_j} = \frac{\partial W(\alpha_i)}{\partial u_j} = W'(\alpha_i)\frac{\partial \alpha_i}{\partial u_j} + \frac{\partial W}{\partial u_j}(\alpha_i) = \frac{\partial W}{\partial u_j}(\alpha_i) \tag{4.14}$$

implies that

$$\frac{\partial W}{\partial u_j}(z) = \frac{\partial a_{n-1}}{\partial u_j} z^{n-1} + \cdots + \frac{\partial a_0}{\partial u_j} + \frac{\partial v_1/\partial u_j}{z - v_{m+1}} + \cdots + \frac{v_m}{(z - v_{m+1})^{m+1}} \frac{\partial v_{m+1}}{\partial u_j} \tag{4.15}$$

can be rewritten as

$$\frac{\partial W}{\partial u_j}(z) = \frac{\prod_{k=1, j\neq k}^{n+m+1} (z - \alpha_k)}{(z - v_{m+1})^{m+1}} \frac{(\alpha_j - v_{m+1})^{m+1}}{\prod_{k=1, j\neq k}^{n+m+1} (\alpha_j - \alpha_k)}. \tag{4.16}$$

Consider

$$g\left(\frac{\partial W}{\partial u_i}, \frac{\partial W}{\partial u_j}\right) = \mathrm{Res}_{z\in \mathrm{Ker} W'}\left(\frac{(\partial W/\partial u_i)(\partial W \partial u_j)}{W'}\right) \mathrm{d}z. \tag{4.17}$$

Recalling (4.12) and (4.16) we find that $g(\partial W/\partial u_i, \partial W/\partial u_j) = 0$ for $i \neq j$. For $i = j$, we find

$$g\left(\frac{\partial W}{\partial u_i}, \frac{\partial W}{\partial u_i}\right) = \mathrm{Res}_{z\in \mathrm{Ker} W'}\left(\frac{(\partial W/\partial u_i)^2}{W'}\right) \mathrm{d}z = \frac{(\alpha_i - v_{m+1})^{m+1}}{\prod_{j=1, j\neq i}^{n+m+1} (\alpha_i - \alpha_j)} = \frac{\partial a_{n-1}}{\partial u_i} \tag{4.18}$$

where the last identity was obtained by comparing coefficients of the z^{n-1} term in (4.15) and (4.16).

Hence, in terms of the coordinates u_i the metric can be rewritten as $g = \sum_{i=1}^N h_i^2(u)(\mathrm{d}u_i)^2$ with the Lamé coefficients

$$h_i^2(u) = \frac{\partial a_{n-1}}{\partial u_i}. \tag{4.19}$$

The fact that $h_i^2(u)$ is a gradient ensures that the rotation coefficients β_{ij} are symmetric and therefore the metric becomes the Darboux–Egoroff metric when expressed in terms of the orthogonal curvilinear coordinates u_i.

5. $N = 3$ models, examples of rational Landau–Ginsburg models

5.1. $n = m = 1$ *model*

Consider the model with $n = m = 1$:

$$W(z) = \frac{1}{2}z^2 + x_1 + \frac{x_2}{z - x_3} \tag{5.1}$$

where as coefficients we used the flat coordinates $x_1 = -\tilde{x}_1$ and x_2, x_3 corresponding to x_1, x_2 of the previous section. The flat coordinates x_α, $\alpha = 1, 2, 3$ are related to the flat metric

$$\eta^{\alpha\beta} = \eta_{\alpha\beta} = \mathrm{Res}_{z\in \mathrm{Ker} W'}\left(\frac{(\partial W/\partial x_\alpha)(\partial W/\partial x_\beta)}{W'}\right) \mathrm{d}z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{5.2}$$

The metric tensor can be derived from the more general expression involving the structure constants

$$c^{\alpha\beta\gamma} = \mathrm{Res}_{z\in \mathrm{Ker} W'}\left(\frac{(\partial W/\partial x_\alpha)(\partial W/\partial x_\beta)(\partial W/\partial x_\gamma)}{W'}\right) \mathrm{d}z \tag{5.3}$$

through relation $\eta^{\alpha\beta} = c^{\alpha\beta 1}$. The non-zero values of the components of $c_{\alpha\beta\gamma}$ are found from (5.3) to be

$$c^{111} = 1 \qquad c^{123} = 1 \qquad c^{222} = 1/x_2 \qquad c^{233} = x_3 \qquad c^{333} = x_2 \tag{5.4}$$

the other values can be derived using that $c^{\alpha\beta\gamma}$ is symmetric in all three indices. These values can be reproduced from the formula (2.29) with the prepotential

$$F(x_1, x_2, x_3) = \tfrac{1}{6}x_2(x_3)^3 + \tfrac{1}{6}(x_1)^3 + x_1x_2x_3 + \tfrac{1}{2}(x_2)^2\left(\log x_2 - \tfrac{3}{2}\right). \tag{5.5}$$

The prepotential satisfies the quasi-homogeneity relation (2.33) with $d_F = 3$ with respect to the Euler vectorfield

$$E = x_1\frac{\partial}{\partial x_1} + \frac{3}{2}x_2\frac{\partial}{\partial x_2} + \frac{1}{2}x_3\frac{\partial}{\partial x_3} = x^1\frac{\partial}{\partial x^1} + \frac{1}{2}x^2\frac{\partial}{\partial x^2} + \frac{3}{2}x^3\frac{\partial}{\partial x^3}. \tag{5.6}$$

We now adopt a general discussion of canonical coordinates to the case $n = m = 1$. Let α_i, $i = 1, 2, 3$ be roots of the polynomial (4.3), which in the present case is $W'(z) = z - x_2/(z - x_3)^2$. So, α_i satisfy $W'(\alpha_i) = 0$ or $\alpha_i(\alpha_i - x_3)^2 - x_2 = 0$ for all $i = 1, 2, 3$.

Then, it follows by taking derivatives of $\alpha_i(\alpha_i - x_3)^2 = x_2$ with respect to x_2, x_3 that

$$\frac{\partial\alpha_i}{\partial x_3} = \frac{2\alpha_i}{3\alpha_i - x_3} \qquad \frac{\partial\alpha_i}{\partial x_2} = \frac{1}{(\alpha_i - x_3)(3\alpha_i - x_3)} \tag{5.7}$$

and further that

$$\frac{\partial u_i}{\partial x_3} = \frac{x_2}{(\alpha_i - x_3)^2} = \alpha_i \qquad \frac{\partial u_i}{\partial x_2} = \frac{1}{\alpha_i - x_3} \tag{5.8}$$

for the canonical coordinates $u_i = W(\alpha_i) = \frac{1}{2}\alpha_i^2 + x_1 + x_2/(\alpha_i - x_3)$. We now present a method of inverting the derivatives in (5.8) or alternatively to find the matrix elements m_{ij} of the matrix M from relation (2.23). The sum of the canonical coordinates is equal to $\sum_{i=1}^{3} u_i = 3x_1 + x_3^2$ and therefore

$$1 = 3\frac{\partial x_1}{\partial u_i} + 2x_3\frac{\partial x_3}{\partial u_i} = h_i^2\left(3 + 2x_3\frac{\partial u_i}{\partial x_2}\right) \tag{5.9}$$

where we used the fact that

$$\frac{\partial x_3}{\partial u_i} = m_{i1}^2\frac{\partial u_i}{\partial x_2} \tag{5.10}$$

because of

$$\frac{\partial x_\alpha}{\partial u_i} = m_{i1}m_{i\alpha} \qquad \frac{\partial u_i}{\partial x_\alpha} = \eta_{\alpha\beta}\frac{m_{i\beta}}{m_{i1}} \qquad h_i^2 = m_{i1}^2. \tag{5.11}$$

Hence, from relation (5.9) it holds that $h_i^2 = \left(3 + 2x_3\frac{\partial u_i}{\partial x_2}\right)^{-1}$ or by using equation (5.8) that

$$\frac{\partial x_1}{\partial u_i} = h_i^2 = \frac{\alpha_i - x_3}{3\alpha_i - x_3}. \tag{5.12}$$

Plugging the last equation into equation (5.10) and using relation (5.8) we obtain

$$\frac{\partial x_3}{\partial u_i} = \frac{1}{3\alpha_i - x_3}. \tag{5.13}$$

Similarly, from

$$\frac{\partial x_2}{\partial u_i} = m_{i1}^2\frac{\partial u_i}{\partial x_3} \tag{5.14}$$

we obtain

$$\frac{\partial x_2}{\partial u_i} = \frac{x_2}{(\alpha_i - x_3)(3\alpha_i - x_3)} = \frac{\alpha_i(\alpha_i - x_3)}{(3\alpha_i - x_3)}. \tag{5.15}$$

Furthermore,

$$\frac{\partial \alpha_i}{\partial u_j} = \frac{\partial \alpha_i}{\partial x_2}\frac{\partial x_2}{\partial u_j} + \frac{\partial \alpha_i}{\partial x_3}\frac{\partial x_3}{\partial u_j} \tag{5.16}$$

gives for $i \neq j$:

$$\frac{\partial \alpha_i}{\partial u_j} = \frac{1}{(3\alpha_i - x_3)(3\alpha_j - x_3)}\left(\frac{\alpha_j(\alpha_j - x_3)}{(3\alpha_i - x_3)} + 2\alpha_i\right) \tag{5.17}$$

and for $i = j$:

$$\frac{\partial \alpha_i}{\partial u_i} = \frac{3\alpha_i}{(3\alpha_i - x_3)^2}. \tag{5.18}$$

Using (5.12) and (5.17) we find the rotation coefficients defined in (2.31) to be

$$\beta_{ij} = -\frac{(\alpha_k - x_3)(3\alpha_k - x_3)}{(3\alpha_i - x_3)(3\alpha_j - x_3)}\frac{1}{\sqrt{(\alpha_i - x_3)(3\alpha_i - x_3)(\alpha_j - x_3)(3\alpha_j - x_3)}}. \tag{5.19}$$

Its square is then

$$\beta_{ij}^2 = -\frac{1}{(\alpha_i - \alpha_j)^2}\frac{1}{(4x_3 - 3\alpha_k)^2}\frac{\partial x_1}{\partial u_k} \tag{5.20}$$

where i, j, k are cyclic. Recall that in equation (3.1) we have introduced the functions $\omega_k = (u_j - u_i)\beta_{ij}$, where again we used the cyclic indices i, j, k. The difference of canonical coordinates can be written as: $u_j - u_i = (\alpha_i - \alpha_j)(3\alpha_k - 4x_3)/2$ which together with equation (5.19) yields

$$\omega_k^2 = -\frac{1}{4}h_k^2 = -\frac{1}{4}\frac{\partial x_1}{\partial u_k} = -\frac{1}{4}\frac{\alpha_k - x_3}{3\alpha_k - x_3}. \tag{5.21}$$

Since $I = \sum_{i=1}^{3} \partial/\partial u_i = \partial/\partial x_1$ then

$$\sum_{k=1}^{3}\omega_k = -\frac{1}{4} \qquad E(\log\tau) = \frac{1}{4}. \tag{5.22}$$

The explicit form of the roots α_i is needed to find expressions for ω_k and its dependence on the parameter s. It is convenient to introduce $q = x_2/(x_3)^3$ and $a_i = \alpha_i/x_3$ which satisfy equation $a_i(a_i - 1)^2 = q$. Let us furthermore introduce a parameter ω such that $q = 4(\omega^2 - 1)^2/(\omega^2 + 3)^3$. This parametrization makes it possible to obtain the compact expressions for ω_k. The three solutions to the algebraic equation

$$a(a-1)^2 = q = 4\frac{(\omega^2 - 1)^2}{(\omega^2 + 3)^3} \tag{5.23}$$

are

$$a_1 = \frac{4}{\omega^2 + 3} \qquad a_2 = \frac{(\omega + 1)^2}{\omega^2 + 3} \qquad a_3 = \frac{(\omega - 1)^2}{\omega^2 + 3}. \tag{5.24}$$

Note that $a_2 \leftrightarrow a_3$ under $\omega \leftrightarrow -\omega$ transformation, which shows that ω is a purely imaginary variable. First, we find that the variable s from (3.2) can be expressed as

$$s = \frac{(a_2 - a_1)}{(a_3 - a_1)}\frac{(3a_3 - 4)}{(3a_2 - 4)} = \frac{(\omega - 3)^3(\omega + 1)}{(\omega + 3)^3(\omega - 1)}. \tag{5.25}$$

Next, from relations $h_i^2 = (a_i - 1)/(3a_i - 1)$ and equation (5.21) we derive

$$\omega_1^2 = -\frac{1}{4}\frac{(\omega^2 - 1)}{(\omega^2 - 9)} \qquad \omega_2^2 = \frac{1}{4}\frac{(\omega + 1)}{\omega(\omega - 3)} \qquad \omega_3^2 = -\frac{1}{4}\frac{(\omega - 1)}{\omega(\omega + 3)}. \tag{5.26}$$

They provide solutions to the Euler top equations (3.3)–(3.5). The corresponding function [16, 17]

$$y(\omega) = \frac{(\omega-3)^2(\omega+1)}{(\omega+3)(\omega^2+3)} \tag{5.27}$$

connected with ω_k through relations [13–15]:

$$\begin{aligned}
\omega_1^2 &= -\frac{(y-s)y^2(y-1)}{s}\left(v-\frac{1}{2(y-s)}\right)\left(v-\frac{1}{2(y-1)}\right)\\
\omega_2^2 &= \frac{(y-s)^2y(y-1)}{s(1-s)}\left(v-\frac{1}{2(y-1)}\right)\left(v-\frac{1}{2y}\right)\\
\omega_3^2 &= -\frac{(y-s)y(y-1)^2}{(1-s)}\left(v-\frac{1}{2y}\right)\left(v-\frac{1}{2(y-s)}\right)
\end{aligned} \tag{5.28}$$

with the auxiliary variable v defined by the equation

$$\frac{\mathrm{d}y}{\mathrm{d}s} = \frac{y(y-1)(y-s)}{s(s-1)}\left(2v-\frac{1}{2y}-\frac{1}{2(y-1)}+\frac{1}{2(y-s)}\right) \tag{5.29}$$

is a solution of the Painlevé VI equation [13–15]

$$\begin{aligned}
\frac{\mathrm{d}^2y}{\mathrm{d}s^2} &= \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-s}\right)\left(\frac{\mathrm{d}y}{\mathrm{d}s}\right)^2-\left(\frac{1}{s}+\frac{1}{s-1}+\frac{1}{y-s}\right)\frac{\mathrm{d}y}{\mathrm{d}s}\\
&\quad+\frac{y(y-1)(y-s)}{s^2(s-1)^2}\left[\frac{1}{8}-\frac{s}{8y^2}+\frac{s-1}{8(y-1)^2}+\frac{3s(s-1)}{8(y-s)^2}\right].
\end{aligned} \tag{5.30}$$

Introducing parameter $x = (\omega-3)/(\omega+3)$ one can rewrite expressions (5.27) and (5.25) as

$$y = \frac{x^2(x+2)}{x^2+x+1} \qquad s = \frac{x^3(x+2)}{2x+1} \tag{5.31}$$

which reproduces the $k = 3$ Poncelet polygon solution of Hitchin [14, 15]. Note also that taking an inverse of solution (5.27) and letting $\omega \to -\omega$ produces the $k = 6$ Poncelet polygon solution of Hitchin [14, 15]:

$$y^{-1}(-\omega) = \frac{(\omega-3)(\omega^2+3)}{(\omega-1)(\omega+3)^2} = \frac{x(x^2+x+1)}{2x+1}. \tag{5.32}$$

We now proceed to calculate the underlying τ function. Our knowledge of the τ function is based on equation (2.19) from which we derive that

$$\partial_j \log\tau = \sum_{i=1}^{3}\beta_{ij}^2(u_i-u_j). \tag{5.33}$$

The identity $I(\log\tau) = 0$ shows that $\tau = \tau(x_2, x_3)$ is a function of two variables x_2, x_3. Furthermore, it satisfies

$$E(\log\tau) = \left(\frac{3}{2}x_2\frac{\partial}{\partial x_2}+\frac{1}{2}x_3\frac{\partial}{\partial x_3}\right)\log\tau = \frac{1}{4}. \tag{5.34}$$

A solution to the above equation is

$$\log\tau = \tfrac{1}{4}\left(\tfrac{1}{3}\log x_2+\log x_3\right)+f\left(\tfrac{1}{3}\log x_2-\log x_3\right) \tag{5.35}$$

where $f(\cdot)$ is an arbitrary function of its argument. In order to determine the function f we use equation (5.33) to calculate the derivative

$$\frac{\partial \log \tau}{\partial x_3} = \sum_{j=1}^{3} \frac{\partial u_j}{\partial x_3} \partial_j \log \tau = \sum_{i,j=1}^{3} \alpha_j \beta_{ij}^2 (u_i - u_j). \tag{5.36}$$

A calculation based on equation (5.20) yields

$$x_3 \frac{\partial}{\partial x_3} \log \tau = \frac{1}{8} \frac{1}{1 - \frac{27}{4} q} = \frac{1}{4} - f' \left(\frac{1}{3} \log x_2 - \log x_3 \right) \tag{5.37}$$

where the last equality was obtained by comparing with equation (5.35) (recall that $q = x_2/(x_3)^3$). Integration gives (ignoring an inessential integration constant)

$$f\left(\tfrac{1}{3} \log x_2 - \log x_3\right) = \tfrac{1}{24} (\log q + \log(-4 + 27q)). \tag{5.38}$$

Using that $x_2 = q x_3^3$ we can now rewrite $\log \tau$ as

$$\log \tau = \tfrac{1}{4} \log x_3^2 + \tfrac{1}{24} \log(q^3(-4 + 27q)). \tag{5.39}$$

Inserting parametrization of q from (5.23) and using relation $u_2 - u_3 = 8x_3^2 \omega^3 (\omega^2 + 3)^{-2}$ we obtain the following expression for $\log \tau$:

$$\log \tau = \log(u_2 - u_3)^{\frac{1}{4}} + \tfrac{1}{24} \log((\omega - 1)^6 (\omega + 1)^6 (\omega - 3)^2 (\omega + 3)^2 \omega^{-16}). \tag{5.40}$$

It is easy to confirm $I(\log \tau) = 0$ and $E(\log \tau) = 1/4$ based on this expression.

5.2. $n = 0$, $m = 2$ *model*

Consider the model with $n = 0, m = 2$ in (4.1):

$$W(z) = z + \frac{x_1}{z - x_3} + \frac{x_2^2}{2(z - x_3)^2}. \tag{5.41}$$

The flat coordinates $x_\alpha, \alpha = 1, 2, 3$ are related to the flat anti-diagonal metric

$$\eta^{\alpha\beta} = \eta_{\alpha\beta} = \mathrm{Res}_{z \in \mathrm{Ker} W'} \left(\frac{(\partial W/\partial x_\alpha)(\partial W/\partial x_\beta)}{W'} \right) \mathrm{d}z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{5.42}$$

The model is characterized by the prepotential

$$F = \tfrac{1}{2} x_3^2 x_1 + \tfrac{1}{2} x_2^2 x_3 + \tfrac{1}{2} x_1^2 \log(x_2) \tag{5.43}$$

which generates the structure constants according to relation (2.29) and possesses homogeneity $d_f = 4$ with respect to the Euler operator

$$E = 2x_1 \frac{\partial}{\partial x_1} + \frac{3}{2} x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} = x^1 \frac{\partial}{\partial x^1} + \frac{3}{2} x^2 \frac{\partial}{\partial x^2} + 2x^3 \frac{\partial}{\partial x^3}. \tag{5.44}$$

Plugging the values $d_1 = 1, d_2 = 3/2, d_3 = 2, d_F = 4$ into the relation (3.9) we find that the homogeneity of $\log \tau$ is again equal to $R^2 = 1/4$. The roots $\alpha_i, i = 1, 2, 3$ of $W'(\alpha_i) = 0$ with W given in (5.41) satisfy the equation

$$(\alpha - x_3)^3 - (\alpha - x_3) x_1 - x_2^2 = 0. \tag{5.45}$$

It is convenient to introduce instead variables $f_i = \alpha_i - x_3$ which satisfy

$$f^3 - f x_1 - x_2^2 = 0. \tag{5.46}$$

Clearly, f_i are functions of x_1 and x_2 only, and by taking derivatives of (5.46) we obtain

$$\frac{\partial f_i}{\partial x_1} = \frac{f_i}{3 f_i^2 - x_1} \qquad \frac{\partial f_i}{\partial x_2} = \frac{2x_2}{3 f_i^2 - x_1}. \tag{5.47}$$

In terms of variables f_i the canonical coordinates become

$$u_i = W(\alpha_i) = x_3 + \frac{3}{2} f_i + \frac{1}{2} \frac{x_1}{f_i} \tag{5.48}$$

and satisfy $\sum_{i=1}^{3} u_i = 3x_3 - x_1^2 / 2x_2^2$. We find that the Lamé coefficients are given by

$$h_i^2 = \frac{\partial x_1}{\partial u_i} = \frac{f_i^3}{3f_i^2 - x_1}. \tag{5.49}$$

The homogeneity of h_i^2 is found after applying a general formula $E\partial_i = \partial_i E - \partial_i$ to the above equality with the result

$$E\big(h_i^2\big) = E\left(\frac{\partial x_1}{\partial u_i}\right) = \partial_i E(x_1) - \partial_i x_1 = (d_3 - 1)\frac{\partial x_1}{\partial u_i} = (d_3 - 1)h_i^2 = h_i^2. \tag{5.50}$$

Here we used that $d_3 = \sum_{\alpha=1}^{3} d_\alpha \eta^{\alpha 1} = 2$ for the model under consideration. Thus the current Lamé coefficients h_i^2 will not depend on only one variable s. Note that in the previous $n = m = 1$ model the value $d_1 = \sum_{\alpha=1}^{3} d_\alpha \eta^{\alpha 1} = 1$ was consistent with homogeneity of h_i^2 being zero.

The corresponding rotation coefficients

$$\beta_{ij} = \frac{1}{2\sqrt{\frac{\partial x_1}{\partial u_i}\frac{\partial x_1}{\partial u_j}}} \frac{\partial^2 x_1}{\partial u_i \partial u_j} \tag{5.51}$$

are calculated straightforwardly from knowledge of (5.49) and following derivatives

$$\frac{\partial f_i}{\partial u_j} = \frac{2x_2^2 f_j + f_j^3 f_i}{\big(3f_i^2 - x_1\big)\big(3f_j^2 - x_1\big)} \qquad i \neq j. \tag{5.52}$$

They are explicitly given by

$$\beta_{ij} = \frac{1}{\sqrt{f_i f_j}} \frac{1}{\sqrt{3f_k^2 - 4x_1}} \frac{N_k}{(f_i - f_j)^2 \big(3f_k^2 - x_1\big)^{(5/2)}} \tag{5.53}$$

where $N_k = 8x_2^4 f_k^2 - 4x_1 x_2^4 + f_i^4 f_j^4$ with indices i, j, k being cyclic.

The result for $\omega_k = (u_j - u_i)\beta_{ij}$ is

$$\omega_k^2 = \frac{r}{4} \frac{3g_k^2 - 4}{(g_k - 3r)\big(3g_k^2 - 1\big)^5} \left(8g_k^4 - 4g_k^2 + \frac{r^2}{g_k^2}\right)^2 \tag{5.54}$$

where $g_k, k = 1, 2, 3$ are three roots of the equation $g^3 - g - r = 0$ with $r = x_2^2 / x_1^{3/2}$.

One verifies that ω_k^2 from (5.54) do indeed satisfy $\sum_{k=1}^{3} \omega_k^2 = -1/4$ and the Euler top equations (3.3)–(3.5). This model provides an example of the Lamé coefficients with non-zero homogeneity and consequently ω_k^2 are not proportional to the Lamé coefficients.

This model allows for another class of solutions of the Euler top equation. Consider, namely

$$\tilde{h}_i^2 = \frac{\partial x_3}{\partial u_i} = \frac{f_i^2}{3f_i^2 - x_1} \tag{5.55}$$

with the properties $\sum_{i=1}^{3} \tilde{h}_i^2 = 1$ and $E\big(\tilde{h}_i^2\big) = (d_1 - 1)\tilde{h}_i^2 = 0$.

The corresponding rotation coefficients

$$\tilde{\beta}_{ij} = \frac{1}{2\sqrt{\frac{\partial x_3}{\partial u_i}\frac{\partial x_3}{\partial u_j}}} \frac{\partial^2 x_3}{\partial u_i \partial u_j} \tag{5.56}$$

are found to be given by

$$\tilde{\beta}_{ij} = \frac{x_2^2}{2} \frac{3f_k^2 - x_1}{\left(3f_i^2 - x_1\right)^{(3/2)} \left(3f_j^2 - x_1\right)^{(3/2)}} \tag{5.57}$$

which produces $\tilde{\omega}_k = (u_j - u_i)\tilde{\beta}_{ij}$ equal to

$$\tilde{\omega}_k^2 = -\frac{1}{16}\tilde{h}_k^2 = -\frac{1}{16} \frac{f_k^2}{\left(3f_k^2 - x_1\right)} \tag{5.58}$$

which satisfy $\sum_{k=1}^{3} \omega_k^2 = -1/16$ and the Euler top equations (3.3)–(3.5).

6. Discussion

This paper shows how to derive the canonical integrable structures for some class of rational Lax functions associated with a particular reduction of the dispersionless KP hierarchy. This derivation generalizes the well-known construction of the monic polynomials [3, 12]. The three-dimensional examples provide solutions to the Painlevé VI equation. Given that the flows of the canonical integrable models can essentially be reformulated as isomonodromic deformations, its connection to the sixth Painlevé equation is not surprising.

Much of the discussion is centred around the τ functions from which all the objects of the Darboux–Egoroff metric can be derived. The τ functions of the three-dimensional examples had a scaling dimension of $R^2 = 1/4$ and the corresponding prepotentials contained logarithmic terms. For the scaling dimensions, $R^2 = n^2$ such that n is an integer, the multi-component KP hierarchy provides a framework for the construction of canonical integrable hierarchies. The long-term goal of this work is to search for a universal approach to the formulation of the canonical integrable models which would include models with fractional scaling dimensions such as the ones encountered in examples based on the rational potentials of the LG type.

Further progress is needed for the classification of relevant rational reductions of the Toda and KP hierarchies and the canonical integrable models with associated Darboux–Egoroff metrics which can be derived from related rational Lax functions. This problem is currently under investigation.

Acknowledgments

HA was partially supported by FAPESP and NSF (PHY-9820663). AHZ and JFG were partially supported by CNPq. The authors thank Johan van de Leur for reading the manuscript and his comments on it.

References

[1] Witten E 1990 On the structure of the topological phase of two-dimensional gravity *Nucl. Phys.* B **340** 281

[2] Dijkgraaf R, Verlinde H and Verlinde E 1991 Topological strings in $D < 1$ *Nucl. Phys.* B **352** 59

[3] Dubrovin B 1995 Geometry of 2D topological field theories *Springer Lect. Notes Math.* **1620** 120 (*Preprint* hep-th/9407018)

[4] Dubrovin B 1992 Integrable systems in topological field theory *Nucl. Phys.* B **379** 627

[5] van de Leur J 2001 Twisted GL_n loop group orbit and solutions of the WDVV equations *Int. Math. Res. Not* **11** 551 (*Preprint* nlin.si/0004021)

[6] Aratyn H, Gomes J F and Zimerman A H 2001 Multidimensional Toda equations and topological-anti-topological fusion *J. Geom. Phys.* at press (*Preprint* hep-th/0107056)

[7] van de Leur J W and Martini R 1999 The construction of Frobenius manifolds from KP tau-functions *Commun. Math. Phys.* **205** 587 (*Preprint* solv-int/9808008)

[8] Aratyn H and van de Leur J 2003 Integrable structure behind WDVV equations Contribution to *NEEDS'01 Conference in Cambridge, Theor. Math. Phys.* **134** 12

[9] Aratyn H and van de Leur J 2001 Solutions of the WDVV equations and integrable hierarchies of KP type *Preprint* hep-th/0104092

[10] Aratyn H and van de Leur J in preparation

[11] Aoyama S and Kodama Y 1996 Topological Landau–Ginzburg theory with a rational potential and the dispersionless KP hierarchy *Commun. Math. Phys.* **182** 185 (*Preprint* hep-th/9505122)

Aoyama S and Kodama Y 1994 Topological conformal field theory with a rational W potential and the dispersionless KP hierarchy *Mod. Phys. Lett.* A **9** 2481 (*Preprint* hep-th/9404011)

[12] Hitchin N J 1995 Frobenius manifolds. With notes by David Calderbank *Gauge Theory and Symplectic Geometry (NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. vol 488)* Montreal PQ pp 69–112

[13] Hitchin N J 1995 Twistor spaces, Einstein metrics and isomonodromic deformations *J. Diff. Geom.* **42** 30

[14] Hitchin N J 1996 Poncelet polygons and the Painlevé transcendents *Geometry and Analysis* (Bombay: Oxford University Press) pp 151–85

[15] Hitchin N J 1996 *A New Family of Einstein Metrics, Manifolds and Geometry (Symposia Mathematica Series vol 36)* ed De Bartolomeis, Triceri and Vesentini (Cambridge: Cambridge University Press)

[16] Segert J 1996 Painlevé solutions from equivariant holomorphic bundles *Preprint* (available from www.math.missouri.edu/~jan/papers/painpreprint.pdf)

[17] Segert J 1998 Frobenius manifolds from Yang–Mills instantons *Math. Res. Lett.* **5** 327 (*Preprint* dg-ga/9710031)

[18] Dubrovin B and Zhang Y 2001 Normal forms of hierarchies of integrable PDEs, Frobenius manifolds and Gromov–Witten invariants *Preprint* math.dg/0108160

[19] Chang J H and Tu M H 2001 Topological field theory approach to the generalized Benney hierarchy *Preprint* hep-th/0108059

[20] Strachan I A B 1999 Degenerate Frobenius manifolds and the bi-Hamiltonian structure of rational Lax equations *J. Math. Phys.* **40** 5058 (*Preprint* solv-int/9807004)